Galois Extensions of Structured Ring Spectra

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Conventions

The Hom of an ∞-category will always be a space, while Map will always mean a the mapping spectra of a stable category, with some module structure equipped if necessary. Moreover, colimits and limits are always homotopy colimits and limits.
S is the sphere spectrum.

Conditions on topological groups

Definition

Recall that for a spectrum X, its dual is defined to be $\mathrm{Map}(X, S)$, where S is the sphere spectrum. We define a topological group G to be stably dualizable if its suspension spectrum $S[G] := \Sigma^\infty_+ G$ is dualizable. We denote for simplicity its dual to be DG_+.

Recall

Recall for a ring spectrum R, a right R-module A and a left R-module B, $A \otimes_R B$ as

$$\varinjlim\left(A \otimes B \leftrightarrows A \otimes R \otimes B \quad \cdots \right)$$

and for a left R-module A and a left R-module B, $\mathrm{Hom}_R(A, B)$ is

$$\varprojlim\left(\mathrm{Map}(A, B) \rightleftarrows \mathrm{Map}(A \otimes R, B) \; \cdots \; \mathrm{Map}(A \otimes R \otimes R, B) \right)$$

Proposition

$S[G]$ admits a ring spectrum structure, and for a spectrum X, a left $S[G]$-module structure on X is equivalent to a left G-action on X, and if we equip S with the trivial bimodule structure, we have $S \otimes_{S[G]} X = X_{hLG}$, and $\mathrm{Map}_{LS[G]}(S, X) = X^{hLG}$. We have a similar theorem for right $S[G]$-module structures on X. Likewise, we define for an $\mathbb{E}_\infty$-ring spectrum A, the group algebra $A[G] = A \otimes S[G]$.

proof

The ring spectrum structure of $S[G]$ comes from the property that Σ_+^∞ is symmetric monoidal, allowing us to inherit a ring structure from the group structure of G.
Then, given a $S[G]$-module structure on X, the $g \in G$ acts on X by the map $X \otimes S \hookrightarrow X \otimes S[G] \to X$, where $S \hookrightarrow S[G]$ is given by the inclusion $\{g\} \hookrightarrow G$, and the second morphism the structure map of the module.
Therefore we have equivalences of categories $\mathrm{Fun}(BG, \mathbf{Sp}) \simeq {}_{S[G]}\mathbf{Mod}$.

proof (continued)

Then, notice that given a spectrum X, the trivial G-action on X is given by the module structure $S[G] \otimes X \to X$ given by the projection of all elements of G to a point. In particular, the trivial action of G on S is given this way. We conclude that this functor is equivalent to associating a spectrum X the constant X-valued BG-diagram in $\mathbf{Sp}$. Moreover, We see that $S \otimes_{S[G]} X$ and $\mathrm{Map}_{LS[G]}(S, X)$ are by definition the left adjoint and right adjoint of the inclusion via trivial modules. By the uniqueness of the left and right adjoint, we see that $S \otimes_{S[G]} X = X_{hLG}$ and $\mathrm{Map}_{LS[G]}(S, X) = X^{hLG}$.

Corollary

DG_+ admit a right G-action, with the right $S[G]$-module sturcture given by the dual of the left $S[G]$-module structure on $S[G]$, explicitly given by the following

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{Sp}}(\mathrm{Map}_{\mathbf{Sp}}(S[G], S) \otimes S[G], \mathrm{Map}_{\mathbf{Sp}}(S[G], S)) \\ \simeq &\mathrm{Hom}_{\mathbf{Sp}}(\mathrm{Map}_{\mathbf{Sp}}(S[G], S) \otimes S[G] \otimes S[G], S) \end{aligned}$$

and the map is the preimage of

$$\mathrm{Map}_{\mathbf{Sp}}(S[G], S) \otimes (S[G] \otimes S[G]) \to \mathrm{Map}_{\mathbf{Sp}}(S[G], S) \otimes S[G] \to S$$

Definition

Notice that $S[G]$ admits a Hopf algebra structure with coalgebra structure given by the diagonal map $\Delta : G \to G \times G$, the counit given by the map collapsing G to a point, and the conjugation given by the inverse map $i : G \to G$.

Definition

For a stable dualizable group, we define the dualizing spectrum $S^{\mathrm{ad}G} = S[G]^{hRG}$, where $S[G]$ is regarded as a right G-module.

Theorem

We have $DG_+ \otimes S^{\mathrm{ad}G} \simeq S[G]$.

proof

We consider the shear map given by

$$DG_+ \otimes S[G] \xrightarrow{\mathrm{id}\otimes\Delta} DG_+ \otimes S[G] \otimes S[G] \xrightarrow{m\otimes\mathrm{id}} DG_+ \otimes S[G]$$

We claim that this map has an inverse given by

$$\begin{aligned} & DG_+ \otimes S[G] \xrightarrow{\mathrm{id}\otimes\Delta} DG_+ \otimes S[G] \otimes S[G] \\ \xrightarrow{\mathrm{id}\otimes i\otimes\mathrm{id}} & DG_+ \otimes S[G] \otimes S[G] \xrightarrow{m\otimes\mathrm{id}} DG_+ \otimes S[G] \end{aligned}$$

where the middle map is induced by the inverse map on the middle $S[G]$-factor.

proof (continued)

Then, notice that if we take a right G-action on the left given by the right G-action on $S[G]$, the shear map takes this G-action to the standard right G-actions. Thus taking homotopy fixed points on both sides, we obtain on the left $(DG_+ \otimes S[G])^{hRG} \simeq DG_+ \otimes S^{\mathrm{ad}G}$, and on the right we have

$$\begin{aligned}
&(DG_+ \otimes S[G])^{hRG} \\
\simeq &\mathrm{Map}_{\mathbf{Sp}}(S[G], S[G])^{hRG} \\
\simeq &\mathrm{Map}_R G(S, \mathrm{Map}_{\mathbf{Sp}}(S[G], S[G])) \\
\simeq &\mathrm{Map}_R G(S[G], S[G]) \\
\simeq &S[G]
\end{aligned}$$

Corollary

Dually we define $S^{-\mathrm{ad}G} = (DG_+)_{hLG}$. Moreover, we have $S[G] \otimes S^{-\mathrm{ad}G} \simeq DG_+$.

Corollary

This implies $S^{\mathrm{ad}G} \otimes S^{-\mathrm{ad}G} \simeq S$.

proof

From the above we deduce that $S[G] \otimes S^{\mathrm{ad}G} \otimes S^{-\mathrm{ad}G} \simeq S[G]$. Then by shearing the G actions on both sides to only acting on $S[G]$ if necessary, taking G-homotopy orbits on both sides we obtain the result.

Theorem

For X a left G-module, this map induces the norm map $N : (X \otimes S^{\mathrm{ad}G})_{hLG} \to X^{hLG}$. As a result, we may define the Tate construction X^{tG} to be the cofiber of the norm map.

proof

Consider the limit-colimit exchange map

$$\kappa : ((X \otimes S[G])^{hRG})_{hLG} \to ((X \otimes S[G])_{hLG})^{hRG}$$

where $X \otimes S[G]$ is equipped with the right action acting only on $S[G]$, and the left action acting diagonally.

Lemma

We have $(S[G]\otimes X)^{tG}\simeq *$.

proof

Notice the source of the norm map can be identified with $(S[G]\otimes X\otimes S^{\mathrm{ad}G})_{hLG}\simeq X\otimes S^{\mathrm{ad}G}$, since we may always shear the G-action on the left to the action only on $S[G]$. Then on the right we have

$$\begin{aligned}(S[G]\otimes X)^{hRG}&\simeq(DG_+\otimes S^{\mathrm{ad}G}\otimes W)^{hRG}\\ \simeq&\mathrm{Map}(S[G],W\otimes S^{\mathrm{ad}G})^{hRG}\simeq W\otimes S^{\mathrm{ad}G}\end{aligned}$$

Then we obtain the result.

Definitions and examples

Definition

Let $A \to B$ be a map of $\mathbb{E}_\infty$ ring spectra, and let G be a stably dualizable group acting on the left B through $\mathbb{E}_\infty$ A-algebras (this basically implies that $A \to B$ is also a map of left G-modules). We say that this is a G-Galois extension if we have $A \to B^{hG}$ is an equivalence, and $h : B \otimes_A B \to \mathrm{Map}(S[G], B)$ is an equivalence, where this second map is obtained from the dual $B \otimes_A B \otimes S[G] \to B \otimes_A B \to B$.

Lemma

For any $\mathbb{E}_\infty$-ring spectrum A with G stably dualizable, the map $A \to \mathrm{Map}(S[G], A)$ is a G-Galois extension. The map $A \to \mathrm{Map}(S[G], A)$ is the dual of the collapse map mapping G to $*$.

Theorem

Recall that the Adams operation $\psi^{-1} : KU \to KU$ is the operation sending a vector bundle to its complex conjugate, and $(\psi^{-1})^2 = \mathrm{id}$. Then let $c : KO \to KU$ be the complexification, then we have c and ψ^{-1} exhibits KU as a C_2-Galois extension of KO.

Dualizability and faithfulness

Definition

Recall for A an $\mathbb{E}_\infty$-algebra, an A-module M is faithful if for any other A-module N we have $M \otimes_A N = 0 \implies N = 0$. We say an A-algebra B is faithful over A if it is faithful as an A-module.

Lemma

Let $A \to B$ be a map of $\mathbb{E}_\infty$-algebras, and M a faithful A-module, we have the following.

1. A map of A-modules $X \to Y$ is a weak equivalence iff $M \otimes X \to M \otimes Y$ is a weak equivalence.
2. $B \otimes_A M$ is a faithful B-module.
3. If $A \to B$ is moreover faithful, and for some A-module N we have $B \otimes_A N$ is a faithful B-module we have N is a faithful A-module.

Lemma

Let $A \to B$ be a map of $\mathbb{E}_\infty$-algebras, and M an A-module. Then we have the following.

1. If M is dualizable as an A-module, then $B \otimes_A M$ is a dualizable B-module.
2. If $A \to B$ is faithful, B dualizable over A, and $M \otimes_A B$ is a dualizable B-module, then M is a dualizable A-module.

Lemma

Let $A \to B$ be a map of $\mathbb{E}_\infty$-ring spectra, and let G be a stably dualizable group acting on B through A-algebra homomorphisms. Then if M is a dualizable A-module, then the canonical map $M \otimes_A B^{hG} \simeq (M \otimes_A B)^{hG}$ is an equivalence.

proof

Notice that the left hand side could be identified with $\mathrm{Map}(D_A M, B^{hG})$, while the right hand side could be identified with $\mathrm{Map}_A(D_A M, B)^{hG}$, and we may simply use the fact that taking limits commutes with the second factor of Map.

Proposition

Let A and B be $\mathbb{E}_\infty$ ring spectra, and B be a G-Galois extension of A, then we have

1. For each B-module M a natural equivalence
$$h_M : M \otimes_A B \to \mathrm{Map}_{\mathbf{Sp}}(S[G], M)$$
2. For each B-module M a natural equivalence
$$j_M : M \otimes S[G] \to \mathrm{Map}_A(B, M)$$

Proposition

Let $A \to B$ be a G-Galois extension, then B is a dualizable A-module.

proof

We show the map $\mathrm{Map}_A(B, A) \otimes B \to \mathrm{Map}_A(B, B)$ is an equivalence. Note that this is the top map in the following diagram, with the second B on the left and the right acually replaced by arbitrary A-modules.

proof (continued)

$$\begin{array}{ccc}
M \otimes_A \mathrm{Map}_A(B, A) & \longrightarrow & \mathrm{Map}_A(B, M \otimes_A A) \\
\downarrow & & \downarrow \\
M \otimes_A \mathrm{Map}_A(B, B^{hLG}) & \longrightarrow & \mathrm{Map}_A(B, M \otimes_A B^{hLG}) \\
\downarrow & & \downarrow \\
M \otimes_A \mathrm{Map}(B, B)^{hLG} & \longrightarrow & \mathrm{Map}_A(B, M \otimes_A B)^{hLG} \\
\uparrow & & \uparrow \\
M \otimes_A (B \otimes S[G])^{hLG} & \longrightarrow & (M \otimes_A B \otimes S[G])^{hLG} \\
\uparrow & & \uparrow \\
M \otimes_A (B \otimes S[G] \otimes S^{\mathrm{ad}G})_{hRG} & \to & (M \otimes_A B \otimes S[G] \otimes S^{\mathrm{ad}G})_{hRG}
\end{array}$$

Proposition

A G-Galois extension $A \to B$ is faithful iff the norm map $N : (B \otimes S^{\mathrm{ad}G})_{hG} \to B^{hG}$ is an equivalence.

proof

If the norm map is an equivalence, and M is an A-module such that $M \otimes_A B \simeq *$, then we have the following.

$$M \simeq M \otimes_A A \simeq M \otimes_A B^{hG} \simeq M \otimes_A (B \otimes S^{\mathrm{ad}G})_{hG}$$
$$\simeq (M \otimes_A \otimes_B \otimes S^{\mathrm{ad}G})_{hG} \simeq *$$

Thus $A \to B$ is faithful.

proof (continued)

For the converse, it suffices to prove $B \otimes_A (B \otimes S^{\mathrm{ad}G})_{hG} \to B \otimes_A B^{hG}$ is an equivalence. To prove this, consider the diagram

$$\begin{tikzcd}
B \otimes_A (B \otimes S^{\mathrm{ad}G})_{hLG} \arrow[r] \arrow[d] & B \otimes_A B^{hRG} \arrow[d] \\
(B \otimes_A B \otimes S^{\mathrm{ad}G})_{hLG} \arrow[r] \arrow[d] & (B \otimes_A B)^{hRG} \arrow[d] \\
(\mathrm{Map}(S[G], B) \otimes S^{\mathrm{ad}G})_{hLG} \arrow[r] & \mathrm{Map}(S[G], B)^{hRG}
\end{tikzcd}$$

Notice that all the vertical maps are equivalences by preceding lemmas or definitions, and the bottom horizontal map is an equivalence by a dual argument of the fact that the norm map is an equivalence on free G-modules.

Proposition

Let $A \to B$ be a map of $\mathbb{E}_\infty$ ring spectra, and let G be a stably dualizable group acting on B through A-algebra maps. Then $A \to B$ is a faithful G-Galois extension iff $B \otimes_A B \to \mathrm{Map}(S[G], B)$ is an equivalence and B is faithful and dualizable over A.

proof

$\implies$ is obvious, and we prove $\impliedby$. We show $A \to B^{hG}$ hence $B \to B \otimes_A B^{hG}$ is an equivalence. But we have $B \simeq \mathrm{Map}(S[G], B)^{hG} \simeq (B \otimes_A B)^{hG} \simeq B \otimes_A B^{hG}$ where the last map is again by the dualizability of B.

Proposition

Let $A \to B$ be a G-Galois extension with G an $\mathbb{E}_\infty$ group. Then B is smash invertible as an $A[G]$-module iff $A \to B$ is faithful.

Lemma

Let $A \to B$ be a map of $\mathbb{E}_\infty$ ring spectra, and let $A \to C$ be a G-Galois extension. Then we have the following.

1. If $A \to C$ is faithful, then $B \to B \otimes_A C$ is a faithful G-Galois extension.
2. If B is dualizable over A, then $B \to B \otimes_A C$ is a G-Galois extension.

Lemma

Let $A \to B$ and $A \to C$ be maps of $\mathbb{E}_\infty$ ring spectrum, and suppose B is faithful and dualizable over A. Then let G be a stably dualizable group acting on C through A-algebra maps. Then we have $B \to B \otimes_A C$ is a (faithful) G-Galois extension implies $A \to C$ is a (faithful) G-Galois extension.

The Galois correspondence

Definition

Let G be a stably dualizable group, and we say a subgroup $K \subseteq G$ is allowable if K is stably dualizable, the collapse map $G \times_K EK \to G/K$ is a stable equivalence, and the projection $G \to G/K$ admits a section. We say K is an allowable normal subgroup if furthermore K is normal. Finally we define the equivalences between allowable subgroups to be the minimal equivalence relation generated by the inclusions $K \subseteq K'$ and $K \hookrightarrow K'$ is a stable equivalence.

Theorem (Galois correspondence, part I)

Let $A \to B$ be a faithful G-Galois extension and $K \subseteq G$ be any allowable subgroup. Then the inclusion $C = B^{hK} \to B$ is a faithful G-Galois extension. Moreover, if K is an allowable normal subgroup, then $A \to C$ is a faithful G-Galois extension.

proof

We will prove the extensions are Galois by detecting it through the dualizable and faithful algebras. That is, since B is faithful and dualizable over A, we have $C \otimes_A B$ is faithful and dualizable over C. Thus to prove $C \to B$ is Galois, it suffices to prove that

$$B \otimes_A C = C \otimes_C (C \otimes_A B) \to B \otimes_C (C \otimes_A B) = B \otimes_A B$$

is Galois. We consider the commutative diagram

proof (continued)

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B \arrow[d] & B \arrow[l] \arrow[d] \\
C \arrow[r] \arrow[d] & C \otimes_A B \arrow[r] \arrow[d] & \mathrm{Map}(S[G], B)^{hK} \arrow[d] & \mathrm{Map}(\Sigma^\infty_+ G/K, B) \arrow[l] \arrow[d] \\
B \arrow[r] & B \otimes_A B \arrow[r] & \mathrm{Map}(S[G], B) & \mathrm{Map}(S[G], B) \arrow[l]
\end{tikzcd}$$

we must prove that all the horizontal maps on the middle and right are equivalences, so that the extension $B \otimes_A C \to B \otimes_A B$ would be equivalent to the extension $\mathrm{Map}(\Sigma^\infty_+ G/K, B) \to \mathrm{Map}(S[G], B)$. But this is the trivial extension by the allowable condition, as the following shows.

$$\mathrm{Map}(S[G], B) \simeq \mathrm{Map}(\Sigma^\infty_+(K \times G/K), B) \simeq F(S[K], \mathrm{Map}(\Sigma^\infty_+(G/K), B))$$

proof (continued)

Then use the fact that $A \to B$ is faithful and use the previous lemmas we have $C \to B$ is a faithful extension. Now, only the maps in the middle row are non-trivially equivalences, but indeed they are since

$$B \otimes_A C \simeq B \otimes_A B^{hK} \simeq (B \otimes_A B)^{hK} \simeq \mathrm{Map}(S[G], B)^{hK}$$

$$\mathrm{Map}(\Sigma^\infty_+(G/K), B) \simeq \mathrm{Map}(S[G]_{hK}, B) \simeq \mathrm{Map}(S[G], B)^{hK}$$

Finally it remains to treat the case where K is normal. If this is the case, then G/K is also a group, hence the extension $B \to B \otimes_A C$ is equivalent to the trivial extension $B \to \mathrm{Map}(S[G/K], B)$, hence is faithfully G/K-Galois. And since B is faithful and dualizable over A, we may reflect this faithful extension back and obtain that $A \to C$ is faithfully G/K-Galois.

Definition

We define a category $\mathcal{C}$ to be Grothendieck prestable if there is some presentable stable category $\mathcal{D}$ equipped with a t-structure compatible with filtered colimits, and $\mathcal{C} \simeq \mathcal{D}_{\geq 0}$. In fact, we can canonically find such $\mathcal{D}$ by taking $\mathcal{D} = \mathbf{Sp}(\mathcal{C})$, and associate it with the t-structure coming from $\mathcal{C}$.
If furthermore $\mathcal{C}$ is endowed with a symmetric monoidal structure, we require the above definitions to be compatible with our symmetric monoidal structure.

Definition

Notice for such $\mathcal{C}$, we have doing truncations and taking connective covers agreeing with the t-structure of $\mathcal{D}$. Then we define $\mathcal{C}^{\heartsuit}$ by its discrete objects, and we have $\mathcal{C}^{\heartsuit} \simeq \mathbf{Sp}(\mathcal{C})^{\heartsuit}$. Then we define for $X \in \mathcal{C}$, $\pi_i(X) \in \mathcal{C}^{\heartsuit}$ to be $(\Omega_i X)_{\leq 0}$.
Notice that a symmetric monoidal structure on $\mathcal{C}$ gives a symmetric monoidal structure on $\mathcal{C}^{\heartsuit}$ making $\pi_0 : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ symmetric monoidal.

Definition

We define a grading on a symmetric monoidal Grothendieck prestable category $\mathcal{C}$ to be an autoequivalence $[1] : \mathcal{C} \to \mathcal{C}$ requiring $(c \otimes d)[1] \simeq c[1] \otimes d$. As usual, we will denote by its inverse $[-1]$. The autoequivalence preserves Σ and Ω, and hence preserves the t-structure of $\mathcal{C}$. In particular, $[1]$ preserves the subcategory of discrete objects, so $[1]$ restricts to an autoequivalence $\mathcal{C}^\heartsuit \to \mathcal{C}^\heartsuit$, and we have $(\pi_i X)[k] \simeq \pi_i(X[k])$. We define the bigraded Ext groups for objects in $\mathcal{C}^\heartsuit$ by

$$\mathrm{Ext}^{s,t}_{\mathcal{C}^\heartsuit}(A, B) = \mathrm{Hom}_{D(\mathcal{C}^\heartsuit)}(A, \Sigma^s B[-t])$$

viewed as an object in $\mathcal{C}^\heartsuit$. Finally we can always give any Grothendieck prestable category the trivial grading by $[1] = \mathrm{id}_{\mathcal{C}}$.

Definition

We say $\mathcal{C}$ is separable if the homotopy groups detect equivalences, and complete if the Postnikov towers converge, that is, for every $X \in \mathcal{C}$ we have $X \to \varprojlim_n X_{\leq n}$ is an equivalence.

Definition

A shift algebra A is an $\mathbb{E}_1$-algebra in $\mathcal{C}$ equipped with a map $\tau : \Sigma A[-1] \to A$ of right A-modules inducing an isomorphism $\pi_* A \simeq \pi_0 A[\tau]$, where the latter is defined by the graded algebra in $\mathcal{C}^\heartsuit$ by $(\pi_0 A[\tau])_k \simeq (\pi_0 A)[-k]$.

lemma

Let A be a shift algebra, and let M be a left A-module. Then the following are equivalent.

1. $\pi_* M \simeq \pi_* A \otimes_{\pi_0 A} \pi_0 M$.
2. $\pi_0 A \otimes_A M \in \mathcal{C}^{\heartsuit}$.
3. $\tau \otimes \mathrm{id}_M : \Sigma M[-1] \to M$ is a 1-connective cover.

Definition

We define a left A-module M to be periodic if it satisfies the above equivalent conditions. We denote the category of periodic modules to be the full subcategory $\mathbf{Mod}_A^{per}(\mathcal{C}) \subseteq \mathbf{Mod}_A(\mathcal{C})$.

Theorem

Let A be a shift $\mathbb{E}_2$-algebra, then we have $\mathbf{Mod}_A^{per}(\mathcal{C}) \simeq \mathbf{Mod}_A^{\tau^{-1}}(\mathbf{Sp}(\mathcal{C}))$, where the second category is the stable subcategory of $\mathbf{Mod}_A(\mathbf{Sp}(\mathcal{C}))$ consisting of τ-local modules.

const

For any associative algebra A, its postnikov tower is a tower of associative algebras

$$A \to \cdots \to A_{\leq n} \to \cdots \to A_{\leq 0}$$

where the passage from $A_{\leq n-1}$ to $A_{\leq n}$ is obtained by a square zero extension (we just point it out here, and it would be crucial for later purposes). This induces a tower of Grothendieck prestable categories by

$$\mathbf{Mod}_A \to \cdots \to \mathbf{Mod}_{A_{\leq n}} \to \cdots \to \mathbf{Mod}_{A_{\leq 0}}$$

By giving a filtration on each category $\mathbf{Mod}_{A_{\leq n}}$ by truncation, we may prove that $\mathbf{Mod}_A \simeq \varprojlim \mathbf{Mod}_{A_{\leq n}}$. Hence to understand $\mathbf{Mod}_A$, it suffices to understand $\mathbf{Mod}_{A_{\leq 0}}$, and the relations between $\mathbf{Mod}_{A_{\leq n}}$ and $\mathbf{Mod}_{A_{\leq n-1}}$.

Construction

We define $F_n \in \mathbf{Mod}_{A_{\leq n}}$ by the fiber sequences $F_n \to A_{\leq n} \to A_{\leq n-1}$, which gives a pullback diagram of the form

$$\begin{tikzcd} A_{\leq n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A_{\leq n-1} \arrow[d, "d_0"] \\ A_{\leq n-1} \arrow[r, "d"'] & A_{\leq n-1} \oplus \Sigma F_n \end{tikzcd}$$

Here $A_{\leq n-1} \oplus \Sigma F_n$ is a trivial square-zero extension, and d_0 is the trivial section, while d is a map of algebras depending on the square-zero extension $A_{\leq n} \to A_{\leq n-1}$.

Construction

This square induces a pullback square of module categories.

$$\begin{tikzcd}
\mathbf{Mod}_{A_{\le n}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{Mod}_{A_{\le n-1}} \arrow[d, "d_0^*"] \\
\mathbf{Mod}_{A_{\le n-1}} \arrow[r, "d^*"'] & \mathbf{Mod}_{A_{\le n-1}\oplus \Sigma F_n}
\end{tikzcd}$$

here d^* and d_0^* denote the pushforward (tensor) of modules induced by the map d and d_0. Then we let $\Theta : \mathbf{Mod}_{A_{\le n-1}} \to \mathbf{Mod}_{A_{\le n-1}}$ is defined as $\Theta M = d_* d_0^* M$, where d_* is the pullback of modules along d. Explicitly, we have $\Theta M \simeq M \oplus \Sigma^{n+1}(A_{\le 0} \otimes_{A_{\le n-1}} M)$.
Then consider the projection $p : A_{\le n-1} \oplus \Sigma F_n \to A_{\le n-1}$, noticing that d and d_0 are both sections of p, we have a map $d_* d_0^* M \to d_* p^* p_* d_0^* M \simeq M$. This gives a natural transformation of functors $\pi : \Theta \to \mathrm{id}$.

Theorem

If $\mathbf{Mod}_A$ is generated by periodic modules, then we have an equivalence of categories

$$\mathbf{Mod}_A \simeq D_{\geq 0}(\mathbf{Mod}_{\pi_0 A}(C^{\heartsuit}))$$

Theorem

We define the category of Θ-sections $\Theta - \mathrm{Sect}_{A_{\leq n-1}}$ to be the category of triples (M, s, h), where $M \in \mathbf{Mod}_{A_{\leq n-1}}$, $s : M \to \Theta M$ and h is a homotopy from $\pi \circ s$ to id_M. Then we have $\mathbf{Mod}_{A_{\leq n}} \simeq \Theta - \mathrm{Sect}_{A_{\leq n-1}}$. This means that M can be lifted to $\mathbf{Mod}_{A_{\leq n}}$ iff Θ has a section.

Definition

Recall that a module M is periodic iff $\pi_*(M) \simeq \pi_*(A) \otimes_{\pi_0(A)} \pi_0(M)$. We say that an $A_{\leq n}$-module N is a potential n-stage for a periodic A-module if we have $A_{\leq 0} \otimes_{A_{\leq n}} M$ is discrete. We denote the category of potential n-stages by $\mathcal{M}_n$. Equivalently, for $M \in \mathbf{Mod}_{A_{\leq n}}$ and $n < \infty$, we have M is a potential n-stage iff $\pi_*(M) \simeq \pi_*(A_{\leq n}) \otimes_{\pi_0(A)} \pi_0(M)$. This implies that modules of potential n-stages are n-truncated. Moreover, this implies that the A-module structure on M and $A_{\leq n}$-module structure on M is equivalent.

Lemma

Let $M \in \mathbf{Mod}_{A_{\leq n-1}}$. Then $\pi : \Theta M \to M$ is an n-equivalence.

Corollary

The cofiber of $\pi : \Theta M \to M$ is equivalent to $\Sigma^{n+2}\pi_0 M[-n]$, using the explicit construction of ΘM.

Theorem

Let $M \in \mathcal{M}_{n-1}$, then M could be lifted to a potential n-stage (that is, if there is some $M' \in \mathcal{M}_n$ such that $M \simeq A_{\leq n-1} \otimes_{A_{\leq n}} N$) iff a certain obstruction class $o_M \in \mathrm{Ext}^{n+2,n}_{\pi_0(A)}(\pi_0(M), \pi_0(M))$ vanishes.

proof

Recall that M can be lifted to a potential n-stage iff $\Theta M \to M$ has a section, and by using the cofiber sequence $\Theta M \to M \to \Sigma^{n+2}\pi_0 M[-n]$, the first map admits a section iff the second map is 0. Thus the obstruction to the lifting lies in

$$\begin{aligned} &\mathrm{Hom}_{A_{\leq n-1}}(M, \Sigma^{n+2}\pi_0 M[-n]) \\ \simeq &\mathrm{Hom}_{\pi_0(A)}(\pi_0(A) \otimes_{A_{\leq n-1}} M, \Sigma^{n+2}\pi_0 M[-n]) \end{aligned}$$

using the fact that the target is canonically a $\pi_0(A)$-module. By definition that M is a potential $(n-1)$-stage, $\pi_0(A) \otimes_{A_{\leq n-1}} M \simeq \pi_0(M)$, hence the obstruction lies in

$$\mathrm{Hom}_{\pi_0(A)}(\pi_0(M), \Sigma^{n+2}\pi_0(M)[-n]) \simeq \mathrm{Ext}^{n+2,n}_{\pi_0(A)}(\pi_0(M), \pi_0(M))$$

Theorem

Let $M, N \in \mathcal{M}_n$, then let $uM := A_{\leq n-1} \otimes_{A_{\leq n}} M$, we have a fiber sequence

$$\begin{aligned}\mathrm{Hom}_{\mathcal{M}_n}(M, N) \to \mathrm{Hom}_{\mathcal{M}_{n-1}}(uM, uN) \\ \to \mathrm{Hom}_{D(\mathbf{Mod}_{\pi_0(A)}(C^{\heartsuit}))}(\pi_0(M), \Sigma^{n+1}\pi_0 N[-n])\end{aligned}$$

proof

Notice that $\mathcal{M}_n$ is a full subcategory of $\mathbf{Mod}_A$, moreover $uM \simeq M_{\leq n-1}$, hence the left arrow is equivalent to $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N_{\leq n-1})$. Letting F be the fiber of $N \to N_{\leq n-1}$, we have a fiber sequence

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N_{\leq n-1}) \to \mathrm{Hom}_A(M, \Sigma F)$$

and by noticing $F \simeq \Sigma^n \pi_0 N[-n]$ we obtain the results.

Corollary

We have a spectral sequence $E_1^{s,t} = \mathrm{Ext}^{2s-t,s}_{\pi_0(A)}(\pi_0(M), \pi_0(N))$ converging conditionally to $\pi_*\mathrm{Hom}_A(M, N)$.

proof

Notice that $\mathrm{Hom}_A(M, N) \simeq \varprojlim \mathrm{Hom}_{A_{\leq s}}(M \otimes_A A_{\leq s}, N \otimes_A A_{\leq s})$. But by the previous theorem we have the fiber between the filtrations given by $F_s \simeq \mathrm{Hom}_{D(\pi_0(A))}(\pi_0(M), \Sigma^s \pi_0 N[-s])$. Thus we have a spectral sequence associated to this filtration with the required E_1 page.

Definition

A potential n-stage for an $\mathbb{E}_k$-algebra R in $\mathbf{Mod}_{A\leq n}$ such that its underlying $A_{\leq n}$-module is a potential n-stage. We denote the category of potential n-stages for $\mathbb{E}_k$-algebra by $\mathbf{Alg}_{\mathbb{E}_k}(\mathcal{M}_n)$. This is, of course, equivalent to the condition that $A_{\leq 0} \otimes_{A_{\leq n}} R$ is discrete.
Also by definition $\pi_0(R)$ is an algebra in $\mathbf{Mod}_{\pi_0(A)}(\mathcal{C}^{\heartsuit})$. Finally we have $\mathbf{Alg}_{\mathbb{E}_k}(\mathcal{M}_\infty) \simeq \varprojlim_n \mathbf{Alg}_{\mathbb{E}_k}(\mathcal{M}_n)$.

Lemma

Let R be a potential $(n-1)$-stage for an $\mathbb{E}_k$-algebra. Then the map $\pi : \Theta R \to R$ is a square zero extension of the $\mathbb{E}_k$-$A_{\leq n-1}$-algebra R by the R-module $\Sigma^{n+1}\pi_0 R[-n]$.

Construction

Recall that the $\mathbb{E}_k$-$A_{\leq n-1}$-algebra R has a cotangent complex $\mathbb{L}^{\mathbb{E}_k}_{R/A_{\leq n-1}}$. As usual, we have an equivalence of the mapping space $\mathrm{Hom}_R(\mathbb{L}^{\mathbb{E}_k}_{R/A_{\leq n-1}}, \Sigma M)$ and the space of square zero-extensions of $\mathbb{E}_k$-$A_{\leq n-1}$-algebras $\tilde{R} \to R$ by M. In particular, an extension is trivial iff its corresponding map from the cotangent complex to ΣM is null.

Theorem

Let R be a potential $(n-1)$-stage for an $\mathbb{E}_k$ algebra. Then R can be lifted to a potential n-stage iff some obstruction class

$$o_R \in \mathrm{Ext}^{n+2,n}_{\mathbf{Mod}_{\pi_0(R)}(D(\mathbf{Mod}_{\pi_0(A)}(\mathcal{C}^{\heartsuit})))}(\mathbb{L}^{\mathbb{E}_k}_{\pi_0(R)/\pi_0(A)}, \pi_0(R))$$

vanishes.

Corollary

Let R, T be potential n-stages for an $\mathbb{E}_k$-algebra, and let $uR \to uT$ be the corresponding map in the potential $(n-1)$-stages. Then we have an equivalence

$$F_\varphi \simeq P_{0,\varphi'}\mathrm{Hom}_{\pi_0(R)}(\mathbb{L}^{\mathbb{E}_k}_{\pi_0(R)/\pi_0(A)}, \Sigma^{n+1}\pi_0 T[-n])$$

for F_φ the fiber over φ of the map

$$\mathrm{Hom}_{\mathbf{Alg}_{\mathbb{E}_k}(\mathcal{M}_n)}(R, T) \to \mathrm{Hom}_{\mathbf{Alg}_{\mathbb{E}_k}(\mathcal{M}_{n-1})}(uR, uT)$$

and φ' is a certain morphism in $\mathrm{Hom}_{\pi_0(R)}(\mathbb{L}^{\mathbb{E}_k}_{\pi_0(R)/\pi_0(A)}, \Sigma^{n+1}\pi_0 T[-n])$, where $P_{0,\varphi'}$ denote the path space.

Finally, we have φ (which lives in the potential $(n-1)$-stage) lifts to the potential n-stage iff $\varphi' \simeq 0$. Moreover, the fiber is equivalent to

$$\mathrm{Hom}_{\pi_0(R)}(\mathbb{L}^{\mathbb{E}_k}_{\pi_0(R)/\pi_0(A)}, \Sigma^{n}\pi_0 T[-n])$$

Corollary

Let $\varphi : R \to S$ be a $\mathbb{E}_k$-morphism of periodic A-algebras in $\mathcal{C}$. Then we have a spectral sequence with

$$E_1^{0,0} = \mathrm{Hom}_{\mathbf{Alg}_{\pi_0(A)}}(\pi_0(R), \pi_0(T))$$

$$E_1^{s,t} = \mathrm{Ext}_{\pi_0(R)}^{2s-t,s}(\mathbb{L}_{\pi_0(R)/\pi_0(A)}^{\mathbb{E}_k}, \pi_0(T)), t \geq s > 0$$

converging conditionally to $\pi_{t-s}(\mathrm{Hom}_{\mathbf{Alg}_{\mathbb{E}_k}(\mathbf{Mod}_A)}(R, T), \varphi)$.

Construction

We define an Adams type homology theory E to be a homotopy commutative ring spectrum E which is a filtered colimit $E = \varinjlim E_\alpha$ such that $E_* E_\alpha$ is a finitely generated projective E_*-module, and the Kunneth map $E^* E_\alpha \to \mathrm{Hom}_{E_*}(E_* E_\alpha, E_*)$ is an isomorphism. Then we define $\mathbf{Sp}_E^{fp}$ to be the category of finite spectra with finitely generated projective E-homology, and the covers to be the map inducing E-homology surjections. We define the category of E-synthetic spectra to be $\mathbf{Syn}_E := \mathrm{Sh}_\Sigma^{\mathbf{Sp}}(\mathbf{Sp}_E^{fp})$.

Construction

There is a t-structure on $\mathbf{Syn}_E$. We define for $X \in \mathbf{Syn}_E$, $\pi_n(X)$ to be the sheafification of the presheaf assigning the π_n pointwise. We say X is connective if $\pi_n(X) = 0$ for $n < 0$ and coconnective if $\Omega^\infty X$ is a discrete sheaf of spaces. In particular, we have $\mathbf{Syn}_E^\heartsuit$ the category consisting of sheaves of Abelian groups. This t-structure is right complete. According to our context, we only care about the connective synthetic spectra.

We have a functor $\nu : \mathbf{Sp} \to \mathbf{Syn}_E$ given by the the sheafification of the connective cover of the Yoneda embedding. Moreover, this functor is lax symmetric monoidal.

Finally, we have a bigrading structure on $\mathbf{Syn}_E$ given by $(\Sigma^{s,t} X)(P) = \Sigma^{-t} X(\Sigma^{-s-t} P)$. The categorical suspension is $\Sigma^{1,-1}$.

Construction

We give $\mathbf{Syn}_E^{\geq 0}$ a grading by $[1] = \Sigma^{1,0}$. We claim the following. Firstly, we have for an $\mathbb{E}_\infty$ algebra $A \in \mathbf{Sp}_E$, νA an $\mathbb{E}_\infty$ algebra. We have the following.

1. νA is a shift algebra.
2. $\mathbf{Mod}_{\nu A}^{per} \simeq \mathbf{Mod}_A(Sp_E)$.
3. $\mathbf{Mod}_{\pi_0(\nu A)}(\mathbf{Syn}_E^{\heartsuit}) \simeq \mathbf{Mod}_{E_* A}(\mathbf{Comod}_{E_* E})$.
4. $\mathbf{Mod}_{\nu A}$ is generated by the periodic modules.

We will not give a proof here since we are running out of time. But the above gives rise to our desired spectral sequence.

Theorem

Let $\varphi : R \to T$ be a morphism of E-local $\mathbb{E}_\infty$-A-algebras, then there is a spectral sequence with

$$E_1^{0,0} = \mathrm{Hom}_{\mathbf{CAlg}_{E_*A}(\mathbf{Comod}_{E_*E})}(E_*R, E_*T)$$

$$E_1^{s,t} = \mathrm{Ext}^{2s-t,s}_{\mathbf{Mod}_{E_*R}(\mathbf{CAlg}_{E_*A}(\mathbf{Comod}_{E_*E}))}(\mathbb{L}^{\mathbb{E}_\infty}_{E_*R/E_*A}, E_*T)$$

where E_*T is given E_*R-module structure by the map φ, converging conditionally to

$$\pi_{t-s}(\mathrm{Map}_{\mathbf{CAlg}_A}(R, T), \varphi)$$

Definition

For $A \to B$ a map of $\mathbb{E}_\infty$-ring spectra, we define the topological Hochschild homology of B relative to A by $\mathrm{THH}^A(B) = B \otimes_A S^1$, where this is the tensoring with space in the category $\mathbf{CAlg}(A)$. Equivalently, $\mathrm{THH}^A(B) \simeq B \otimes_{B \otimes_A B} B$. Also equivalently, we have $\mathrm{THH}^A(B)$ is the geometric realization of the cyclic bar construction of B over A. In the last construction, the inclusion into the 0-simplices gives a map $B \to \mathrm{THH}^A(B)$. We say $A \to B$ is THH-etale if $B \to \mathrm{THH}^A(B)$ is an equivalence.

We say $A \to B$ is etale if $\mathbb{L}^{\mathbb{E}_\infty}_{B/A}$ is contractible, B connected if $\pi_0(B)$ is connected, and $A \to B$ separable if $B \otimes_A B \to B$ admits a section.

Lemma

Every separable $A \to B$ is THH-etale. Notice that with G discrete, the G-Galois extension $A \to B$ is separable, hence is THH-etale.

Theorem

For any map $A \to B$ of $\mathbb{E}_\infty$-ring spectra, if it is THH-etale, then it is etale.

Corollary

Any G-Galois extension $A \to B$ with G discrete is etale. In particular, $\prod_G A$ is etale over A.

Theorem

Let $A \to B$ be a G-Galois extension with G finite and B connected. Then the natural map $G \to \mathbf{CAlg}_A(B, B)$ giving the action of G on B is an equivalence of $\mathbb{E}_1$-monoids. In particular, every endomorphism of B over A is an automorphism, up to a contractible choice.

proof

We compute $\mathbf{CAlg}_A(B, B) \simeq \mathbf{CAlg}_B(B \otimes_A B, B) \simeq \mathbf{CAlg}_B(\prod_G B, B)$ by the Goerss-Hopkins spectral sequence derived above. We apply the spectral sequence with the case $E = S$. But since $\prod_G B$ is etale over B, we have the spectral sequence collapses complete at the first page leaving only the term $E_1^{0,0} = \mathbf{CAlg}_{B_*}(\prod_G B_*, B_*) \simeq G$ since B_* is connected. Therefore we have the results.

Theorem

Let $A \to B$ be a G-Galois extension with B connected and G finite and discrete. Then let $A \to C \to B$ be a factorization of the map through a separable $\mathbb{E}_\infty$-A-algebra C. Then $\mathbf{CAlg}_C(B,B)$ is discrete, and through the map $\mathbf{CAlg}_C(B,B) \to \mathbf{CAlg}_A(B,B)$, $\mathbf{CAlg}_C(B,B)$ can be identified with $K = \pi_0(\mathbf{CAlg}_C(B,B))$ as a subgroup of G. Moreover, the action of K on B induces an equivalence $B \otimes_C B \simeq \prod_K B$.

proof

Again using the spectral sequence we see that $\mathbf{CAlg}_C(B,B)$ is discrete, and $K = \pi_0 \mathbf{CAlg}_C(B,B) = \mathbf{Alg}_{\pi_*(B)}(\pi_*(B \otimes_C B), \pi_*(B))$. Then again using the that $\pi_*(B \otimes_C B)$ is a retract of $\pi_*(B \otimes_A B)$, this implies that K is a retract of G, hence is a submonoid of G. But since G is finite, K must also be a group.

Moreover, noticing that $\pi_*(B \otimes_C B)$ is a retract of $\pi_*(B \otimes_A B)$, using the fact that $\pi_*(B)$ is connected, we see that $\pi_*(B \otimes_C B) \simeq \prod_{K'} \pi_*(B)$. Thus $K = K'$, and $B \otimes_A B \simeq \prod_G B$ restricts to $B \otimes_C B \simeq \prod_K B$.

Corollary

Let $A \to B$ be a G-Galois extension with B connected and G finite and discrete. Then let $A \to C \to B$ be a factorization of the map through a separable $\mathbb{E}_\infty$-A-algebra C such that $C \to B$ is faithful, and let $K = \pi_0(\mathbf{CAlg}_C(B, B)) \subseteq G$. If $A \to B$ is faithful, then $C \simeq B^{hK}$ and $C \to B$ is a faithful K-Galois extension.

proof

We first notice $A \to B$ is faithful implies B dualizable over C. Moreover we have $B \otimes_C B \simeq \prod_K B$ is an equivalence, and since B is faithful and dualizable over C, we conclude that $B \to C$ is a faithful K-Galois extension.

Generalizations

Definition

Let A be a $\mathbb{E}_\infty$-ring spectra, and let $A \to B_\alpha$ be a directed family of G_α-Galois extensions, such that for $\alpha \leq \beta$, we have $A \to B_\alpha$ is a subextension of $A \to B_\beta$ equipped with a surjection $G_\beta \to G_\alpha$ with kernel a stably dualizable group $K_{\alpha,\beta}$, and the data of a natural equivalence $B_\alpha \simeq B_\beta^{hK_{\alpha,\beta}}$. Let $B = \varinjlim B_\alpha$ where the colimit is taken in $\mathbb{E}_\infty$-A-algebras, and let $G = \varprojlim G_\alpha$, then we say $A \to B$ is a G-pro-Galois extension.

Example

$L_{K(n)}S \to E_n$ is a $K(n)$-local pro-$\mathbb{G}_n$-Galois extension, where $\mathbb{G}_n$ is the Morava stabilizer group viewed as a profinite group.

Definition

A Hopf algebra spectrum is a $\mathbb{E}_\infty$-ring spectrum H equipped with a counit $\varepsilon : H \to S$ and a co-$\mathbb{E}_1$ coproduct $\psi : H \to H \otimes H$. We do *not* assume that H has an antipode map.
For $A \to B$ a map of $\mathbb{E}_\infty$-ring spectrum, we define the cobar complex $C^\bullet(H; B)$ with H coacting on B over A to be the cosimplicial commutative A-algebra with

$$C^q(H; B) = B \otimes H^{\otimes q}$$

The coboundary and codegeneracy maps are obviously defined, and let $C(H; B)$ denote its totalization. The algebra unit $A \to B$ induces a coaugmentation $A \to C^\bullet(H; B)$, hence a map $A \to C(H; B)$. We say a map of $\mathbb{E}_\infty$ ring spectrum $A \to B$ is an H-Hopf-Galois extension if H coacts on B over A, so that the maps $A \to C(H; B)$ and $h : B \otimes_A B \to B \otimes H$ are both weak equivalences.

Example

For an $\mathbb{E}_\infty$-grouplike space X, $S[X]$ is a $\mathbb{E}_\infty$-ring spectrum through the multilplication of G, and is a Hopf algebra spectrum with counit given by the collapse map $X \to *$ and the comultiplication given by the diagonal map $\Delta : X \to X \times X$.

Example

The unit map $S \to MU$ is an $S[BU]$-Hopf-Galois extension. This can be viewed as the global version of $L_{K(n)}S \to E_n$ is a $K(n)$-local pro-$\mathbb{G}_n$-Galois extension.

Thank you!